Majority-vote model on scale-free hypergraphs

Scaling hypothesis in opinion formation model

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Hypergraphs

Hypergraph

Hypergraph is a generalization of an ordinary graph defined as **H=(V,E)**, where **V** is a set of vertices, and **E** is a set of hyperedges. Each hyperedge is a subset of vertices of any size.
Properties of hypergraphs:

- structure focused on groups
- enables multiparticle interactions

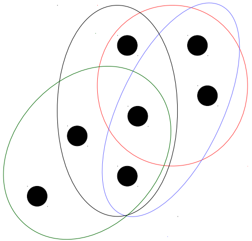


Figure: Simple hypergraph. Hyperedges are represented by lines. Each hyperedge consists of 4 nodes.

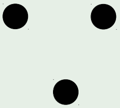
- *Random hypergraphs and their applications*, G. Ghoshal *et al.* PRE **79**, 066118 (2009)
- *Evolving hypernetwork model*, J-W Wang *et al.*, EPJ B **77**, 493 (2010)

Scale-free hypergraphs preferential attachment algorithm

Algorithm parameters

- n - number of nodes in each hyperedge
- m - number of new nodes added in each step
- m_h - number of new hyperedges added in each step

$n = 3,\ m = 2,\ m_h = 2$ example



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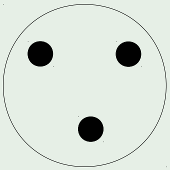

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Hyperedge distribution

$$P(k) \sim k^{-\alpha}$$

$$\alpha = 1 + \frac{n}{n-m}$$

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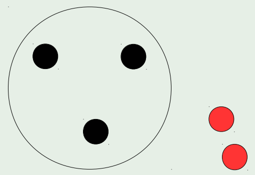

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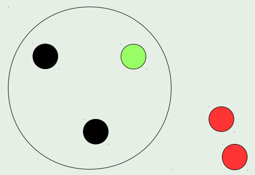

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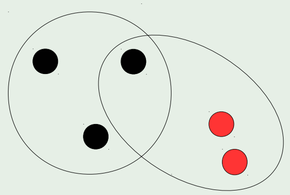

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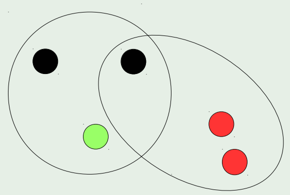

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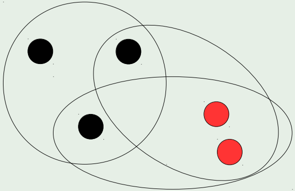

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Majority-vote model

- each node has a spin variable $\sigma_i = \pm 1$
- future state of each spin depends on the majority state of its neighbours
- control parameter q $(0 \leq q \leq 1)$ is introduced, which plays a role of a temperature (noise)
- nodes are updated according to the **hyperedge-update** or **node-update** dynamics
- $S = 0$ gives the *if you don't know what to do, do anything* rule.

M. J. de Oliveira, Journal of Statistical Physics **66** (1992)

Single spin flip probability

$$w_i(\sigma) = \frac{1}{2}\left(1 - (1 - 2q)\sigma_i S\right)$$

where

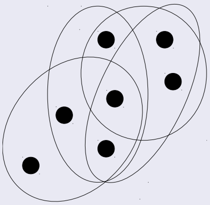
$$S = \mathrm{sgn}\left(\sum_e \sigma_i\right)$$

Sum in S goes over all nodes in selected hyperedge.
sgn(x) is the signum function:

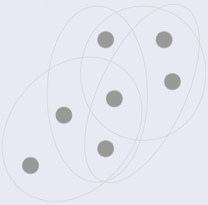
$$sgn(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Hyperedge-update and node-update dynamics

Hyperedge-update dynamics



Node-update dynamics



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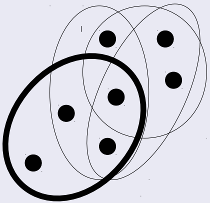

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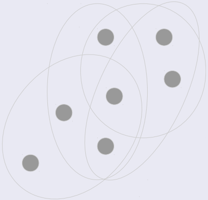

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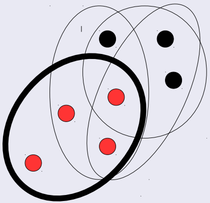

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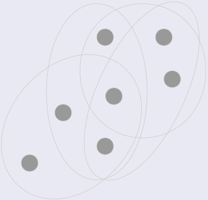

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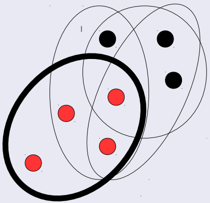

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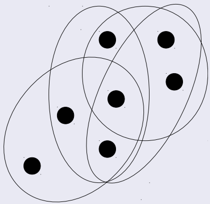

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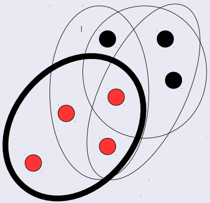

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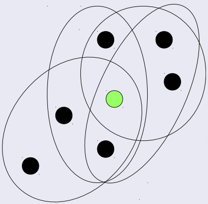

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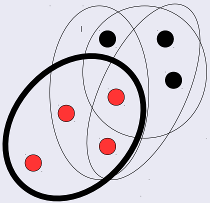

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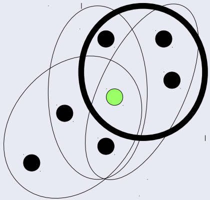

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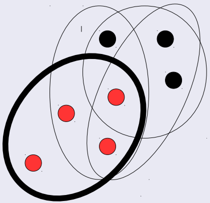

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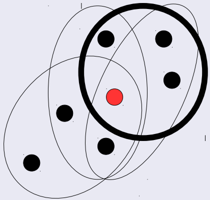

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Simulation parameters

- system size: from $N = 10^3$ up to 10^5
- hypergraph topologies:
  - $n = 3,\ m = 1,\ m_h = 1$; $P(k) \sim k^{-\mathbf{2.5}}$, $|\{e\}| \approx N$
  - $n = 3,\ m = 2,\ m_h = 3$; $P(k) \sim k^{-\mathbf{4}}$, $|\{e\}| \approx \frac{3}{2} N$
  - $n = 4,\ m = 2,\ m_h = 1$; $P(k) \sim k^{-\mathbf{3}}$, $|\{e\}| \approx \frac{1}{2} N$
- averaged over 10 independent realizations of the hypergraph
- transient time: 10^4 steps (hyperedge-update dynamics), 10^5 steps (node-update dynamics)
- measure time: 10^4 steps (hyperedge-update dynamics), 10^5 steps (node-update dynamics)
- at the begining of the simulation all spins are set to *1* (ferromagnetic initial conditions)

Critical phenomena

Magnetization

$$M(q) = [\langle |m| \rangle]_{av}$$

Susceptibility

$$\chi(q) = N\left[\left(\langle m^2 \rangle - \langle m \rangle^2\right)\right]_{av}$$

Reduced fourth-order cumulant (Binder cumulant)

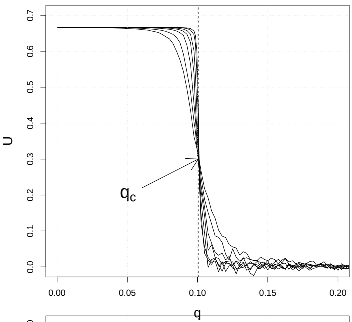
$$U_4(q) = 1 - \left[\frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}\right]_{av}$$

To study the critical behavior we define the variable m:

$$m = \frac{1}{N}\sum_{i=1}^{N}\sigma_i$$

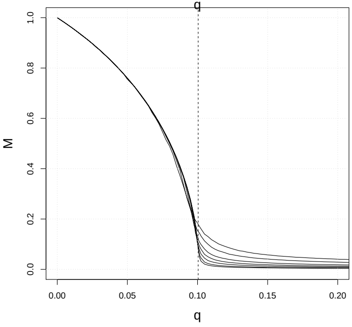
Next, we examine the magnetization M, susceptibility χ and the reduced fourth-order cumulant U_4.

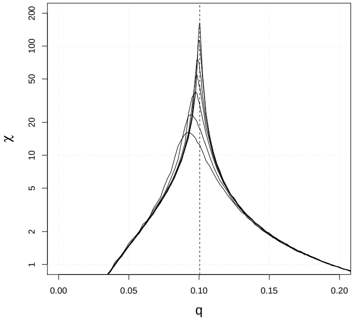
$\langle \ldots \rangle$ stands for the thermodynamic average
$[\ldots]_{av}$ means average over independent realizations of the system (simulations)



Example results: $n = 3$, $m = 2$, $m_h = 3$, hyperedge update dynamics.

The critical noise parameter q_c is estimated as the point where the U_4 curves for different system sizes N intercept each other.





From thermodynamic limit do finite-size scaling

In the thermodynamic limit

$$M \sim (T - T_c)^{\beta}$$

$$\chi \sim (T - T_c)^{-\gamma}$$

$$\xi \sim (T - T_c)^{-\nu}$$

In the finite system of size $N = L^D$ the correlation length $\xi \to L$ in $T = T_c$. Thus:

Finite size scaling

$$M = L^{-\beta/\nu} f_m \left(L^{1/\nu} (T - T_c) \right)$$

$$\chi = L^{\gamma/\nu} f_\chi \left(L^{1/\nu} (T - T_c) \right)$$

Next, we find the critical exponents from the following relations:

$$M(T_c) \sim L^{-\beta/\nu}$$

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Critical exponents obey the Rushbrooke's Identity:

$$\gamma/\nu + 2\beta/\nu = 1$$

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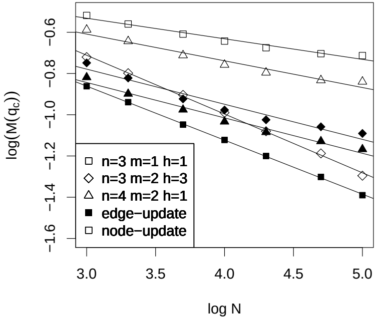
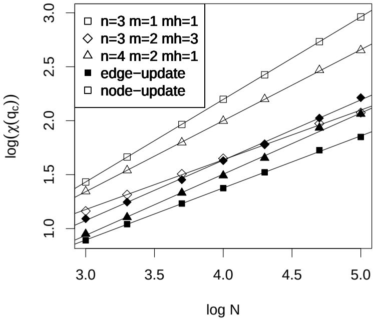
Magnetization and susceptibility scaling

Magnetization



Susceptibility



	hyperedge-update				node-update			
hypergraph	q_c	β/ν	γ/ν	$\gamma/\nu+2\beta/\nu$	q_c	β/ν	γ/ν	$\gamma/\nu+2\beta/\nu$
$n3\ m1\ m_h1$	0.158	0.26	0.48	1.00	0.115	0.10	0.76	0.96
$n3\ m2\ m_h3$	0.100	0.17	0.56	0.90	0.075	0.29	0.46	1.04
$n4\ m2\ m_h1$	0.144	0.17	0.57	0.91	0.102	0.13	0.66	0.92

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- hypergraph is a proper framework for opinion formation model (group is the basic unit)
- hypergraphs enable us to investigate broader class of models
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Raczkowski na koniec